International Journal of Advanced Research and Development
www.advancedjournal.com
ISSN: 2455-4030
Received: 11-07-2022, Accepted: 26-07-2022, Published: 12-08-2022
Volume 7, Issue 4, 2022, Page No. 34-37

Automorphisms of direct and semi-direct product of groups

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Abstract
In this paper, we describe the automorphisms of various classes of finite groups. We introduce the concept of semi direct product on certain groups and use this concept to define the automorphism of certain finite groups. If the automorphism of direct product of two groups is a group then the automorphism groups of these two groups are disjoint. $Aut(T_1 \times T_2) \cong Aut(T_1) \times Aut(T_2)$ iff T_1 as well as T_2 is characteristic subgroup of $T_1 \times T_2$. We compute the number of automorphisms of an abelian $p-$ group of order p^s.

Keywords: automorphisms, direct, semi-direct product

Introduction

The first general structure result for the automorphism group of a finite group follows from a classical result of Gauss in number theory. Let Z_n denotes the additive group of integers $mod\ (n)$. There exist some finite groups that are isomorphic to their own automorphism groups, e.g. D_6. The structure of $Aut(G)$ is often hard to compute. This paper will explore the orders and structures of automorphism groups for direct and semi direct products.

Theorem: Suppose that an abelian group $\boldsymbol{G}$ be such that

$$G \cong \underbrace{Z_p \times Z_p \times \dots \times Z_p}_{s-times}$$

For p –prime and $s \in \mathbb{N}$, then

$$\left|Aut\left(\prod_s Z_p\right)\right| = \prod_{j=0}^{s}(p^s - p^j).$$

Proof: As $G \cong \underbrace{Z_p \times Z_p \times \dots \times Z_p}_{s-times}$, the following cases are:

For $s=1$, in that case $G = Z_p$, there are $p-1$ elements which have order p. Assume that $Z_p = <c>$, suppose $\gamma : Z_p \to Z_p$ defined as $\gamma(c) = c^j$, where $1 \le j \le p-1$ in that case $\gamma \in Aut(Z_p)$. Thus $|Aut(Z_p)| = p-1$.

For $s = 2$, in that case $G = Z_p \times Z_p$, Let $Z_p \times Z_p = <c> \times <d>$. Thus assume $\delta : Z_p \times Z_p \to Z_p \times Z_p$ a homomorphism. In that case $\delta-$automorphism iff $|\delta(c)| = |\delta(d)| = p$ as well as $<\delta(c)> \cap <\delta(d)> = \{e\}$. To find $|Aut(Z_p \times Z_p)|$, the number of pairs $(u,v) \in Z_p \times Z_p$ so that $\delta(c) = u$ as well as $\delta(d) = v$.

$Z_p \times Z_p$ Contains $p^2 - 1$ elements having Order p. As $c \in Z_p \times Z_p$ so that $|c| = p$. So any automorphism sends r to that element of $Z_p \times Z_p$ which also has Order p. And there are $p^2 - 1$ elements in $Z_p \times Z_p$, whose order is p. So there are $p^2 - 1$ choices for image of c having an automorphism. Let $u \ne e$, then we should find the number of element $v \in Z_p \times Z_p$ having order p as well as $<u> \cap <y> = \{e\}$.

A $p-$ group is generated by every u. But there are $p^2 - 1$ elements of order p.

Which implies that $p^2 - p$ elements of order p don't belong to $T = <u>$ and those elements also form a group S of order p whose intersection with $<u>$ is only unit element. Thus $|Aut(Z_p \times Z_p)| = (p^2 - p)(p^2 - 1)$.

For $s \ge 3$, Suppose

$$G = \prod_s Z_p$$

And assume that the set of generators $\{c_1, c_2, \dots, c_s\}$ for G such that $\underbrace{Z_p \times Z_p \times \dots \times Z_p}_{s-times} = <c_1> \times <c_2> \times \dots \times <c_s>$.

Except the identity element, the order of all elements is p. To find $|Aut(G)|$, we should calculate the one-one mapping from generators of G to the groups generating by non-identity element and the intersection contains only identity element. Assume that $\rho(c_1) = r$ for any $\rho \in Aut(G)$. Now there exists $p^s - p$ elements $u \in G$ so that $<r> \cap <u> = \{e\}$. Let assume that $\rho(c_1) = r$ and $\rho(c_2) = t$ for some $t \notin <r>$, hence there exists $p^s - p^2$ elements w

whose order is p so that $w \notin <r> \times <t>$. Now let $\rho(c_3) = w$ such that $(<r> \times <t>) \cap \times <w> = \{e\}$. Continuing like this, we find $|Aut(G)|$ like that

$$\left|Aut\left(\prod_s Z_p\right)\right| = \prod_{j=0}^{s}(p^s - p^j).$$

Theorem: Suppose a group $Aut(T_1 \times T_2)$, then the subgroups $Aut(T_1) \cap Aut(T_2) = \{I\}$, where $I-$ identity of $Aut(T_1 \times T_2)$.
Proof: Assume that a map $\nu : Aut(T_1) \to Aut(T_1 \times T_2)$ defined by $\nu(\mu) = \nu\mu$ for some $\mu \in Aut(T_1)$, where $\nu\mu \in Aut(T_1 \times T_2)$ that is defined by $\nu\mu(t_1, t_2) = (\mu(t_1), t_2)$. Again we define a map $\theta : Aut(T_2) \to Aut(T_1 \times T_2)$ by $\theta(\tau) = \theta\tau$, where $\theta\tau(t_1, t_2) = (t_1, \tau(t_2)) \forall \tau \in Aut(T_2)$. Then ν and θ are 1-1 homomorphisms and hence $Aut(T_1) \cap Aut(T_2) = \{I\}$, where $I-$ identity of $Aut(T_1 \times T_2)$.

Theorem: Suppose $T_1 \times T_2$ be the direct product of T_1 and T_2, then there is a subgroup of $Aut(T_1 \times T_2)$ which is isomorphic to $Aut(T_2) \times Aut(T_1)$ having only those mapping from $Aut(T_1 \times T_2)$ that maps elements from T_1 to T_1 and T_2 to T_2.
Proof: Let $\nu \in Aut(T_1)$ and $\mu \in Aut(T_2)$, then $(\nu, \mu) \in Aut(T_1) \times Aut(T_2)$. Suppose $\theta : T_1 \times T_2 \to T_1 \times T_2$ defined by $\theta(t_1, t_2) = (\nu(t_1), \mu(t_2)) \ \forall \ t_1 \in T_1 \ and \ t_2 \in T_2$. We shall show that $\theta \in Aut(T_1 \times T_2)$. It is very easy to show that θ is well defined as well as 1-1 homomorphism. Now for some $(t_1, t_2) \in T_1 \times T_2$, as both ν and μ are onto therefore we have $(t_1, t_2) = (\nu(t_1'), \mu(t_2')) = \theta(t_1', t_2')$. Therefore $\theta \in Aut(T_1 \times T_2)$. Now define the map $\xi : Aut(T_1) \times Aut(T_2) \to Aut(T_1 \times T_2)$ by $\xi(\nu, \mu) = \theta$. For any $\nu, \nu_1 \in Aut(T_1)$ and $\mu, \mu_1 \in Aut(T_2)$ we have

$$\begin{aligned}
&(\nu, \mu) = (\nu_1, \mu_1) \\
&\Leftrightarrow \nu = \nu_1 \ and \ \mu = \mu_1 \\
&\Leftrightarrow \nu(t_1) = \nu_1(t_1) \ and \ \mu(t_2) = \mu_1(t_2) \\
&\Leftrightarrow (\nu(t_1), \mu(t_2)) = (\nu_1(t_1), \mu_1(t_2)) \\
&\Leftrightarrow \theta(t_1, t_2) = \theta_1(t_1, t_2) \\
&\Leftrightarrow \xi(\nu, \mu)(t_1, t_2) = \xi(\nu_1, \mu_1)(t_1, t_2).
\end{aligned}$$

As this holds for $\forall t_1 \in T_1$ and $t_2 \in T_2$. Thus ξ is 1-1 homomorphism. Hence $Aut(T_1) \times Aut(T_2)$ is subgroup of $Aut(T_1 \times T_2)$ having those elements from $Aut(T_1 \times T_2)$ which maps elements from T_1 to T_1 and T_2 to T_2.

Proposition: $Aut(T_1 \times T_2) \cong Aut(T_1) \times Aut(T_2) \ iff \ T_1$ as well as T_2 is characteristic subgroup of $T_1 \times T_2$.
Proof: Assume that T_1 and T_2 are characteristic subgroups of $T_1 \times T_2$. Then $\forall \ \xi \in Aut(T_1 \times T_2)$ we have $\xi(T_1) \subseteq T_1$ as well as $\xi(T_2) \subseteq T_2$. Thus we get each element belongs to $Aut(T_1 \times T_2)$ sends elements of $T_1 to \ T_1$ as well as elements of $T_2 \ to \ T_2$. From previous result, the subgroup $Aut(T_1) \times Aut(T_2) \subseteq Aut(T_1 \times T_2)$ having those elements from $Aut(T_1 \times T_2)$ which maps elements from T_1 to T_1 and T_2 to T_2. Hence we get $Aut(T_1) \times Aut(T_2) \cong Aut(T_1 \times T_2)$.
Now let assume that $Aut(T_1) \times Aut(T_2) \cong Aut(T_1 \times T_2)$, as each automorphism of $Aut(T_1) \times Aut(T_2)$ sends elements of T_1 to T_1 as well as elements of T_2 to T_2. Thus each automorphism belongs to $Aut(T_1 \times T_2)$ maps elements of $T_1 to \ T_1$ as well as elements of $T_2 \ to \ T_2$, means, $\xi(T_1) \subseteq T_1$ and $\xi(T_2) \subseteq T_2 \ \forall \ \xi \in Aut(T_1 \times T_2)$. Hence both T_1 and T_2 be characteristic in $T_1 \times T_2$.

Proposition: If $(r, s) = 1$, then $Aut(Z_r \times Z_s) \cong Aut(Z_r) \times Aut(Z_s)$.
Proof: Let $(r, s) = 1$. Assume that $Z_r = <c>$ and $Z_s = <d>$, then $Z_r \times Z_s = <c> \times <d>$. Now each automorphism of $Z_r \times Z_s$ can be assumed where it maps c and d. Any automorphism of this type must map c to an element whose order r. Given that $|c^i d^j|$ is equal to $lcm(|c^i|, |d^j|)$, that is equal to $r \ iff \ |c^i| = r \ \& \ |d^j| = 1$, as $(|d|, r) = 1$. Hence, each automorphism of $Z_r \times Z_s$ must map c to some power of c. In the same way all element of $Aut(Z_r \times Z_s)$ must map d to some power of d. Hence we get the automorphisms of $Z_r \times Z_s$ are of the form $\binom{c \mapsto \nu(c)}{d \mapsto \mu(d)}$ for $\nu \in Aut(Z_r)$ and $\mu \in Aut(Z_s)$. Thus we get Z_r and Z_s are characteristic in $Z_r \times Z_s$. Hence $Aut(Z_r \times Z_s) \cong Aut(Z_r) \times Aut(Z_s)$.

Proposition: Assume two finite groups T_1 and T_2 such that $(|T_1|, |T_2|) = 1$, then $Aut(T_1 \times T_2) \cong Aut(T_1) \times Aut(T_2)$.
Proof: Assume two finite groups T_1 and T_2 such that $(|T_1|, |T_2|) = 1$. As it is known that order of group is divisible by order of each of its element. Therefore $(|T_1|, |T_2|) = 1 \ iff \ (|t_1|, |t_2|) = 1 \ \forall \ t_1 \in T_1 \ \& \ t_2 \in T_2$. If $(|t_1|, |t_2|) = 1 \ \forall \ t_1 \in T_1 \ \& \ t_2 \in T_2$. We will prove that T_1 as well as T_2 both is characteristic in $T_1 \times T_2$ by assuming the order of elements belongs to $T_1 \times T_2$. Any automorphism from $Aut(T_1 \times T_2)$ must send every $t_1 \in T_1$ to an element that has order equals to $|t_1|$, since $\forall \ t_2 \in T_2, (|t_1|, |t_2|) = 1$. Therefore given any $t_1' \in T_1$, every non-identity element $t_2 \in T_2$ gives an element $(t_1', t_2) \in T_1 \times T_2$ whose order $\neq |t_1|$. In the same way, given any $t_2' \in T_2$, every non-identity element $t_1 \in T_1$ gives an element $(t_1, t_2') \in T_1 \times T_2$ whose order $\neq |t_2|$. Hence, for given $t_1 \in T_1 \ \& \ t_2 \in T_2$, each automorphism of $T_1 \times T_2$ are of form $(t_1, t_2) \mapsto \nu(t_1)\mu(t_2)$ for some $\nu \in Aut(T_1)$ and $\mu \in Aut(T_2)$. Hence we have $Aut(T_1 \times T_2) \cong Aut(T_1) \times Aut(T_2)$.

Lemma: Assume that G_1, G_2, G_3, G_4 the groups. Let $\nu \in hom(G_3, G_4)$, $\mu \in hom(G_2, G_1)$, and $\delta \in hom(G_2, Aut(G_3))$. Let $\theta: G_3 \rtimes_\delta G_2 \to G_4 \rtimes G_1$ is a function which defined by $\theta(g_3 g_2) = \nu(g_3)\mu(g_2)$ for every element $g_3 \in G_3$ & $g_2 \in G_2$, then θ is homomorphic $iff\ \theta(g_2 g_3) = \theta(g_2)\theta(g_3)\ \forall g_3 \in G_3\ \&\ g_2 \in G_2$.

Proof: Let $\nu \in hom(G_3, G_4)$, $\mu \in hom(G_2, G_1)$, $\delta \in hom(G_2, Aut(G_3))$ and assume that θ is homomorphic. Suppose $\theta(g_2 g_3) \neq \theta(g_2)\theta(g_3)$ for any $g_3 \in G_3$ & $g_2 \in G_2$ then by using definition of θ we get $\mu(g_2)\nu(g_3) \neq \mu(g_2)\nu(g_3)$ a contradiction. Thus $\theta(g_2 g_3) = \theta(g_2)\theta(g_3)\ \forall g_3 \in G_3\ \&\ g_2 \in G_2$.

Now assume that $\theta(g_2 g_3) = \theta(g_2)\theta(g_3)\ \forall g_3 \in G_3\ \&\ g_2 \in G_2$. We shall show that $\theta(rt) = \theta(r)\theta(t)\ \forall r, t \in G_3 \rtimes_\delta G_2$. Without loss of generality suppose that $r = g_3 g_2\ \&\ y = g_3' g_2'$ where $g_3, g_3' \in G_3\ and\ g_2, g_2' \in G_2$. Now we have

$$\begin{aligned}
\theta(rt) &= \theta\big((g_3 g_2)(g_3' g_2')\big)\\
&= \theta(g_3 \delta(g_2) g_3' g_2 g_2')\\
&= \nu\big(g_3 \delta(g_2)(g_3')\big)\mu(g_2 g_2'))\\
&= \nu(g_3)\nu\big(\delta(g_2)(g_3')\big)\mu(g_2 g_2')\\
&= \nu(g_3)\nu\big(\delta(g_2)(g_3')\big)\mu(g_2)\mu(g_2')\\
&= \nu(g_3)\theta(\delta(g_2)(g_3')g_2)\mu(g_2')\\
&= \nu(g_3)\theta(g_2 g_3')\mu(g_2')\\
&= \nu(g_3)\theta(g_2)\theta(g_3')\mu(g_2')\\
&= \nu(g_3)\mu(g_2)\nu(g_3')\mu(g_2')\\
&= \theta(g_3 g_2)\theta(g_3' g_2')\\
&= \theta(r)\theta(s).
\end{aligned}$$

Hence proved.

Theorem: Assume that $\mu \in hom(T_1, Aut(T_2))$. Suppose that $im(\mu)$ is the subgroup of $Z(Aut(T_2))$, the center of the automorphism group of T_2, in that case $Aut(T_2)$ is the subgroup of $Aut(T_2 \rtimes_\mu T_1))$.

Proof: Let $\mu \in hom(T_1, Aut(T_2))$ and $im(\mu)$ – the subgroup of $Z(Aut(T_2))$. Let $\tau: Aut(T_2) \to Aut(T_2 \rtimes_\mu T_1)$ be the function which sends every $\nu \in Aut(T_2)$ to the map $\{t_2 t_1 \mapsto \nu(t_2)t_1\}$ where $t_2 \in T_2$ and $t_1 \in T_1$. Assume that τ_ν represents the $im(\nu)$ through τ, then

$$\begin{aligned}
\tau_\nu(t_1 t_2) &= \tau_\nu(\mu(t_1)(t_2)t_1)\\
&= \nu(\mu(t_1)(t_2)t_1\\
&= \mu(t_1)(\nu(t_2))t_1\\
&= t_1\nu(t_2)\\
&= \tau_\nu(t_1)\tau\nu(t_2),
\end{aligned}$$

Which implies that every τ_ν be the homomorphism. Furthermore, τ is also a homomorphism because $\tau_{\rho\xi} = \tau_\rho\tau_\xi\ \forall \rho, \xi \in Aut(T_2)$. Thus $Aut(T_2) \cong im(\tau)$ that is subgroup of $Aut(T_2 \rtimes_\mu T_1)$.

Theorem: Let $\tau \in hom(T_1, Aut(T_2))$. If $\tau \circ \alpha = \tau\ \forall\ \alpha \in Aut(T_1)$ then $Aut(T_1)$ is subgroup of $Aut(T_2 \rtimes_\tau T_1)$.

Proof: Suppose $\tau \circ \alpha = \tau\ \forall\ \alpha \in Aut(T_1)$, and assume that $\beta: Aut(T_1) \to Aut(T_2 \rtimes_\tau T_1)$ is a function which sends $\alpha \in Aut(T_1)$ to the map $\{t_2 t_1 \mapsto t_2\alpha(t_1)\}$ where $t_2 \in T_2$ and $t_1 \in T_1$. Letting $\beta\alpha$ represents the image of α through β, in that case we get

$$\begin{aligned}
\beta\alpha(t_1 t_2) &= \beta\alpha(\tau(t_1)(t_2)t_1)\\
&= \tau(t_1)(t_2)\alpha(t_1)\\
&= (\tau \circ \alpha)(t_1)(t_2)\alpha(t_1)\\
&= \tau(\alpha(t_1))(t_2)\alpha(t_1)\\
&= \alpha(t_1)t_2\\
&= \beta\alpha(t_1)\beta\alpha(t_2).
\end{aligned}$$

This provides that every $\beta\alpha$ is a homomorphism. Furthermore, β is a homomorphism as

$$\beta\mu\nu = \beta\mu\beta\nu\ \forall\ \mu, \nu \in Aut(T_2).$$

Hence we have $Aut(T_1) \cong im(\beta)$, means, $Aut(T_1)$ is subgroup of $Aut(T_2 \rtimes_\tau T_1)$.

Theorem: Let $(r,s) = 1$, then Z_r is characteristic in $Z_r \rtimes_\tau Z_s\ \forall,\ \tau \in hom(Z_s, Aut(Z_r))$.

Proof: Let $(r,s) = 1$ and for some $\tau \in hom(Z_s, Aut(Z_r))$ we assume

$$Z_r \rtimes_\tau Z_s \cong < s_1, s_2;\ s_1{}^r = s_2{}^s = e, bs_1 = \tau(s_2)(s_1)s_2 >.$$

As $|s_1| = r$, as well as $(r,s) = 1$ hence we get order of $s_1^i s_2^j$ is also r. We expand $(s_1^i s_2^j)^r$ like that

$$(s_1^i s_2^j)^r = \underbrace{(s_1^i\, s_2^j)(s_1^i\, s_2^j)\cdots(s_1^i s_2^j)}_{(r-times)}$$
$$= \left(s_1^i\, s_2^j s_1^i\, (s_2^j)^{-1}\, s_2^j\right)\left(s_2^j\, s_1^i\, (s_2^j)^{-2}\right)(s_2^j)^2 \cdots (s_2^j)^{r-1} s_1^i\, (s_2^j)^{1-r}\, (s_2^j)^{r-1}\, s_2^j$$
$$= \left(s_1^i \tau(s_2^j)(s_1^i)\tau\left((s_2^j)^2\right)(s_1^i) \cdots \tau\left((s_2^j)^{r-1}\right)(s_1^i)\right) s_2^{rj}.$$

Now if $(s_1^i s_2^j)^r = e$ then $s_2^{rj} = e$. Hence, if an automorphism sends s_1 to $s_1^i s_2^j$ then s divides $|s_2^{rj}|$ that is a contradiction because $|s_2^{rj}|$ divides s and
$(r,s) = 1$, Thus we get $j = 0$, which implies for each automorphism $v \in Aut(Z_r \rtimes_\tau Z_s)$ we have $v(s_1) = s_1^i s_2^j = s_1^i\ \forall s_1 \in Z_r$, that means, Z_r is characteristic in $Z_r \rtimes_\tau Z_s$.

Theorem: Let $\rho, \xi \in hom(T_1, Aut(T_2))$. If $\rho \sim \xi$ then $T_2 \rtimes_\rho T_1 \cong T_1 \rtimes_\xi T_2$.
Proof: Since every element in $T_2 \rtimes_\rho T_1$ can be written as the product $t_2 t_1$ for some $t_2 \in T_2$ and $t_1 \in T_1$. The element $t_2 t_1$ also belongs to $T_1 \rtimes_\xi T_2$ so care must be taken while choosing the elements of $T_2 \rtimes_\rho T_1$ and $T_1 \rtimes_\xi T_2$. Suppose that $\rho \sim \xi$ and $\tau \in Aut(T_1)$ satisfying $\rho = \xi \circ \tau$. Define the function $\Phi : T_2 \rtimes_\rho T_1 \to T_1 \rtimes_\xi T_2$ defined by $\Phi(t_2 t_1) = t_2 \tau(t_1)$. Now for each $t_2 t_1 \in T_1 \rtimes_\xi T_2$ we have $t_2 t_1 = t_2(id(t_1)) = t_2(\tau\tau^{-1}(t_1)) = \Phi(t_2 \tau^{-1}(t_1))$, that is, Φ is surjective. Since order of $T_2 \rtimes_\rho T_1$ and $T_1 \rtimes_\xi T_2$ is same. Therefore Φ is also injective. Hence Φ is bijection. Now by Lemma 4.5.7, we show Φ is homomorphism. Observe that $\rho(t_1) = (\xi \circ \tau)(t_1) = \xi(\tau(t_1))\ \forall t_1 \in T_1$, so we have $\Phi(t_1 t_2) = \Phi(\rho(t_1)(t_2)t_1) = \xi(t_1)(t_2)\tau(t_1) = \xi(\tau(t_1))(t_2)\tau(t_1) = \tau(t_1)t_2 = \Phi(t_1)\Phi(t_2)\ \forall t_2 \in T_2\ \&\ t_1 \in T_1$. Hence Φ is isomorphism, that is, $T_2 \rtimes_\rho T_1 \cong T_1 \rtimes_\xi T_2$.

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