International Journal of Advanced Research and Development
ISSN: 2455-4030; Impact Factor: RJIF 5.24
Received: 06-06-2019; Accepted: 10-07-2019
www.advancedjournal.com
Volume 4; Issue 5; September 2019; Page No. 102-104

Subspace Properties, T_0-Identification Spaces, and Weakly Po

Charles Dorsett
Retired Associate Professor, Department of Mathematics, Texas A&M University, Commerce, Texas, USA

Abstract
In this paper each of subspace properties, T_0-identification spaces, and weakly Po spaces and properties are further investigated, and additional connections between the properties are given.

Keywords: subspace properties, T_0-identification spaces, weakly Po spaces and properties

1. Introduction

Preliminaries: Topological subspace properties have been long studied in mathematics. Given below is the classical definition of subspace properties.

Definition 1.1. A topological property P is a subspace property iff for each space with property P, each subspace of the space has property P.

In 1936 [11], T_0-identification spaces were introduced and used to jointly characterize pseudo metrizable and metrizable.

Definition 1.2. Let (X,T) be a space, let R be the equivalence relation on X defined by xRy iff Cl({x}) = Cl({y}), let X_0 be the set of R equivalence classes of X, let $P:X\rightarrow X_0$ be the natural map, and let Q(X,T) be the quotient topology on X_0 determined by (X,T) and the map P. Then $(X_0,Q(X,T))$ is the T_0-identification space of (X,T).

Theorem 1.1. A space is pseudo metrizable iff its T_0-identification space is metrizable.

T_0-identification spaces were cleverly created to add T_0 to an externally generated, strongly (X,T) related T_0-identification space of (X,T), making T_0-identification spaces a strong, useful topological tool. The continued investigation of T_0-identification spaces has not only established additional strong properties for the T_0-identification process, but also revealed new tools that have been used to resolve unaddressed questions in classical topology.

In 2015 [1] pseudo metrizable was generalized to weakly Po.

Definition 1.3. Let P be a topological property for which (P and T_0), which is denoted Po, exists. Then a space has property weakly Po iff its T_0-identification space has property P. A topological property Po for which weakly Po exists is called a weakly Po property.

Since for each space, its T_0-identification space is T_0, then a space is weakly Po iff its T_0-identification space is Po.

In the initial investigation of weakly Po spaces and properties [1], it was shown that weakly Po is a unique topological property simultaneously shared by both a space and its T_0-identification space, and (weakly Po)o = Po.

Based on their definitions and initial uses, subspace properties and weakly Po spaces and properties appeared to be totally independent of each other. However, continued investigation of the three properties showed that not to be the case.

Within the 2015 paper [1], the search for topological properties that are not weakly Po led to the use of T_0 and "not-T_0", where "not-T_0" is the negation of T_0. Thus another fundamental role of T_0 was revealed and "not-T_0" proved to be a useful topological property. In classical topology, "not-P", where "not-P" is the negation of the topological property P that exists, had been almost totally overlooked, but the special, useful role of "not-P" in the study of weakly Po motivated the introduction, investigation, and use of "not-P" as an important, useful topological property in the paper [2], opening a new, never before imagined, fertile topological territory for further investigation and expansion of classical topology.

Our mathematical ancestors did an incredible job in formulating and investigating what today is referred to as modern topology, but there remained natural, unaddressed questions to be resolved, including questions concerning subspaces.

As examples, (1) if P and Q are subspace properties, is (P and Q) a subspace property?, (2) if P is a topological property, does "not-P" exist, and, if not, what about "not-P", where P is a subspace property, (3) are there non-subspace properties other than those given in classical topology, and (4) does every space have a property P that is a subspace property as defined above that destroys continuity in the continued study of subspace and non-subspace properties, and, if so, can the discontinuity be corrected?

With the great number and diversity of the known subspace properties and the tools available for use in classical topology, resolving the questions above would be challenging, if not impossible. However, that situation changed with new properties and tools discovered in the continued study of T_0-identification spaces and the investigation of weakly Po spaces and properties.

Within the paper [2], T_0 and "not-P" were used to give the never before imagined least of all topological properties L.

Theorem 1.2. L = (T_o or "not-T_0") = (P or "not-P"), where P is a topological property for which "not-P" exists, is the least of all topological properties.

Thus, there is a topological property L for which "not-L" does not exist. Are there others?

Within the paper [2], it was shown that every space has property L. Thus, by the definition above, L is a subspace property; far from the intent of subspace properties, creating a discontinuity in the study of subspace properties. In the paper [3], the discontinuity ended with the removal of L as a subspace property.

Definition 1.4. Let P be a topological property. Then P is a subspace property iff P is not L and a space has property P iff every subspace of the space has property P.

If there is a least topological property, is there a strongest topological property? In the paper [4], P and "not-P", where P is a topological property for which "not-P" exists, were used to show there is no strongest topological property. Hence, "not-P", where P is a topological property for which "not-P" exists played yet another important role in answering an unaddressed question in the study of classical topology.

As given above, in the initial use of T_0-identification spaces, a classical topological property P (metrizable) was given and a property Q (pseudo metrizable) was sought such that a space has property Q iff its T_0-identification space has property P, motivating the introduction of weakly Po spaces and properties, as given above. Initially weakly Po spaces and properties were investigated using the same search process. However, such a trial-and-error search process would be tedious and none ending, raising the global question: "For what topological properties P does weakly Po exist?", leading to the resolution of another unaddressed question in classical topology.

Theorem 1.3. {Qo: Qo is a weakly Po property} = {Qo: Q is a topological property and Qo exists} [5].

Thus great progress was made in the study of T_0-dentification spaces, and the uncertainty of which topological properties are weakly Po properties was replaced by certainty and the uncertainty of whether there is a topological property Q for which Q = weakly Po was replaced by certainty.

In the paper [6], properties and discoveries in the investigation of weakly Po spaces and properties were used to show that there are topological properties other than L for which "not-P" does not exist, but, for each subspace property P, "not-P" exists and is a non-subspace property, creating a never before known category of non-subspace properties. In addition, it was shown that for subspace properties P and Q, (P and Q) exists and is a subspace property, quickly and easily creating many new subspace properties, and for subspace properties P and Q for which (P and "not-Q") exists, (P and "not-Q") is a non-subspace property, creating yet another new category of non-subspace properties. Also, in the paper [7], it was shown that the singleton set topological property is the strongest subspace property and, with the removal of L as a subspace property, there is no least subspace property.

Definition 1.5. A space (X,T) has property the singleton set topological property (SSTP) iff X is a singleton set [7].

In classical topology the search for non-subspace properties focused on topological properties P for which there exists a space with property P with a subspace that is not P. Such non-subspace properties will be labeled classical non-subspace properties. An example of a classical non-subspace property is normal. Since there are spaces in which every subspace is normal, within classical topology, completely normal was defined.

Definition 1.6. A space is completely normal iff every subspace of the space is normal.

Within the paper [6], completely P was defined for each classical non-subspace P and it was shown that completely P is a subspace property, creating previously unknown subspace properties.

Definition 1.7. A space is completely P, where P is a classical non-subspace property, iff every subspace of the space has property P.

Hence, the continued investigations of T_0-identification spaces and the investigations of weakly Po spaces and properties have established many new properties and useful connections between subspace properties and the three properties; T_0-identification spaces, and weakly Po spaces and properties, allowing unaddressed questions in classical topology to be resolved. Below, the unaddressed question of whether for a subspace property P, must weakly Po be subspace properties is addressed and resolved, the role of weakly Po for topological and subspace properties P is addressed.

2. Po and Weakly Po for Subspace Properties P.

Since every space has property L, then a space has property L iff its T_0-identification space has property L, which was used in [8] to prove L = weakly Lo = weakly T_0. Since each T_0-identification space is T_0, then for weakly Po to be a subspace property, weakly Po is not L and Po is not T_0.

In the paper [9], the investigation of the behavior of subspaces in the T_0-identification process was investigated, giving the following useful result.

Theorem 2.1. Let (X,T) be a space and let Y be a subset of X. Then $(P(Y),Q(X,T)_{P(y)})$ and $(Y_0,Q(Y,T_Y))$ are homeomorphic.

Theorem 2.2. Let Q be a topological property. Then weakly Qo is a subspace property iff Qo is a subspace property different from T_0.

Proof: Suppose Qo is a subspace property different from T_0. Then, by the results above, weakly Qo exists and is not L. Let (X,T) be weakly Qo. Then $(X_0,Q(X,T))$ has property Qo. Let Y be a non-empty subset of X. Then $(P(Y),Q(X,T)_{P(Y)})$ is a subspace of $(X_0,Q(X,T))$ and has property Qo. Since $(Y_0,Q(Y,T_Y))$ is homeomorphic to $(P(Y),Q(X,T)_{P(Y)),})$, then $(Y_0,Q(Y,T_Y))$ has property Qo and (Y,T_Y) is weakly Qo. Thus every subspace of (X,T) is weakly Qo and weakly Qo is not L, which implies weakly Qo is a subspace property.

Conversely, suppose weakly Qo is a subspace property. Then Qo exists and weakly Qo is not L, which implies Qo is not T_0. Since each of weakly Qo and T_0 are subspace properties, then ((weakly Qo) and T_0)) = (weakly Qo)o = Qo is a subspace property.

Theorem 2.3. Let Q be a subspace property. Then weakly Qo is a subspace property.

Proof: Since P and T_0 are subspace properties, then (P and T_0) exists and is a subspace property, and weakly Po exists.

Also, since Qo is a subspace property Qo is not L. Suppose Qo is T_0. Let (X,T) have property weakly Qo. Then $(X_0,Q(X,T))$ has property Qo. Since a space is T_0 iff the natural map from the space onto its T_0-identification space is a homeomorphism [10], then the T_0-identification space of $(X_0,Q(X,T))$ is T_0 and $(X_0,Q(X,T))$ has property L, which is a contradiction. Thus Qo is not T_0 and by Theorem 2.2, weakly Qo is a subspace property.

In the consideration of the results above, the following questions arose:"If Q is a subspace property, Q not Qo, and a space (X,T) satisfies property Q, must $(X_0,Q(X,T))$ be Qo and/or weakly Qo = Q?" Below the questions are resolved and the rolls of weakly Qo for topological and subspace properties within spaces that are weakly Po are given.

3. Resolution of the Questions and the Roll of Weakly Qo.

Definition 3.1. A space (X,T) has property FinInd iff X is finite and T is the indiscrete topology on X.

The proofs of the next two theorems are straightforward and omitted.

Theorem 3.1. FinInd is a topological property.

Theorem 3.2. FinInd is a subspace property.

Definition 3.2. A space (X,T) has property Ind iff X is a set and T is the indiscrete topology on X.

Theorem 3.3. (FinInd)o = SSTP.

Proof: Let (X,T) have property (FinInd)o. Then X is a singleton set, for suppose not. Let x and y be distinct elements of X. Since (FinInd)o implies Ind, then T is the indiscrete topology on X, Cl({x}) = Cl({y}), and (X,T) is "not-T_0", but then (X,T) is simultaneously "not-T_0" and T_0, which is a contradiction. Thus (FinInd)o = SSTP.

Theorem 3.3. Ind is a topological property.

The straightforward proof is omitted.

Theorem 3.4. Ind = weakly (Ind)o = SSTP.

Proof: Let (X,T) be a space. If (X,T) has property Ind, then for each x in X, the R equivalence class containing x, C_x, equals X and X_0 has property SSTP. Conversely, suppose $(X_0,Q(X,T))$ has property SSTP. Then X_0 is a singleton set. Suppose T is not the indiscrete topology on X. Let O be a proper nonempty open set in X. Let x be in O and let y be in X and not O. Then Cl({x}) is not Cl({y}) and C_x is not C_y, but then X_0 is not a singleton set, which is a contradiction. Thus T is the indiscrete topology on X and (X,T) has property Ind. Hence Ind = weakly (Ind)o = SSTP.

From classical topology, Ind implies pseudo metrizable, which is a subspace property. Since FinInd implies Ind, then FinInd implies pseudo metrizable.

Example 3.1. Let (X,T) have property FinInd. Then (X,T) satisfies the pseudometric property, and, from above, pseudo metrizable = weakly (pseudo metrizable)o = weakly (metrizable). However, from above, (FinInd)o = SSTP, which is stronger than the subspace property metrizable, and Ind = weakly (Ind)o = weakly (SSTP), which is not FinInd.

Theorem 3.5. Let P be a topological property for which weakly Po exists and let *WP* = {weakly Qo: Qo exists and implies Po}. Then weakly Po is the least element of *WP*.

Proof: Let weakly Qo be in *WP*. Let (X,T) be a space with property weakly Qo. Then $(X_0,Q(X,T))$ has property Qo, which implies $(X_0,Q(X,T))$ has property Po and (X,T) has property weakly Po. Since weakly Po exists and Po implies Po, then weakly Po is in *WP*. Thus weakly Po is the least element of *WP*.

Theorem 3.6. Let P be a subspace property and let *P* = {weakly Qo: weakly Qo is a subspace property and Qo implies Po}. Then weakly Po is the least element of *P*.

Proof: Since *P* is a subset of *WP*, then weakly Po is a lower bound for *P* and, by Theorem 2.3, weakly Po is in *P*, then weakly Po is the least element of *P*.

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