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Weyl's theorem for perturbations of paranormal operators

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Abstract
Weyl's theorem for perturbations of paranormal operators is studied by Pietro Aiena and Jesus R. Guillen. In this chapter, if T is a paranormal operator on a Hilbert space, then $T+K$satisfies Weyl's theorem for every algebraic operator Kwhich commutes with T. if T is a paranormal operator on a Hilbert space, then $T+K$ satisfies Weyl's theorem for every algebraic operator Kwhich commutes with T.is discussed.

Keywords: Weyl's theorem, paranormal, Hilbert space, Banach space

1. Introduction: Weyl's theorem and perturbations of paranormal operators

1.1 Theorem [4]

If $T \in L(X)$, then the following asssertions are equivalent:

(i) Weyl's theorem holds for T;

(ii) T has the SVEP at evey point $\lambda \notin \sigma_W(T)$ and $\pi_{00}(T) = p_{00}(T)$.

Let $\mathcal{P}_0(X)$, Xa Banach space, denote the class of all operator $T \in L(X)$

Such that there exists $p = p(\lambda) \in \mathbb{N}$for which

$$H_0(\lambda I - T) = ker(\lambda I - T)^p \text{ For all} \lambda \in \pi_{00}(T).$$

1.2 Theorem

$T \in \mathcal{P}_0(X)$ if and only if $p_{00}(T) = \pi_{00}(T)$.

In particular, if Thas SVEP, then Weyl's theorem holds for Tiff $T \in \mathcal{P}_0(X)$.

Proof

Suppose $T \in \mathcal{P}_0(X)$ and $\lambda \in \pi_{00}(T)$.

Then there exists $p \in \mathbb{N}$ such that

$$H_0(\lambda I - T) = ker(\lambda I - T)^p.$$

Since λ is isolated in $\sigma(T)$, then by theorem 1.3.12

$$\begin{aligned} X &= H_0(\lambda I - T) \oplus K(\lambda I - T) \\ &= ker(\lambda I - T)^p \oplus K(\lambda I - T), \end{aligned}$$

from which we obtain

$$\begin{aligned} (\lambda I - T)^p(X) &= (\lambda I - T)^p(K(\lambda I - T)) \\ &= K(\lambda I - T), \end{aligned}$$

So $X = ker(\lambda I - T)^p \oplus (\lambda I - T)^p(X)$, it follows from theorem 1.3.13 that

$$p(\lambda I - T) = q(\lambda I - T) \le p.$$

By definition of $\pi_{00}(T)$ we known that $\alpha(\lambda I - T) < \infty$

and this implies by theorem 1.3.17 that $\beta(\lambda I - T)$ is also finite.
Therefore $\lambda \in p_{00}(T)$ and hence $p_{00}(T) \subseteq \pi_{00}(T)$.

Since the opposite inclusion holds for every operator we then conclude that

$$p_{00}(T) = \pi_{00}(T).$$

Conversely,

If $p_{00}(T) = \pi_{00}(T)$ and $\lambda \in \pi_{00}(T)$,

then $p := p(\lambda I - T) = q(\lambda I - T) < \infty$.

It follows from theorem 4.3.18 that

$$H_0(\lambda I - T) = ker(\lambda I - T)^p$$

Hence $T \in \mathcal{P}_0(X)$. By theorem 4.2.1

3.3 Theorem
If $T \in L(X)$ is algebraically paranormal, then every isolated point of the spectrum is a pole of the resolvent. Furthermore $T \in \mathcal{P}_0(X)$.

Proof
Note first that every quasi-nilpotent algebraically paranormal operator T is nilpotent.
In fact, Suppose that h is a polynomial for which $h(T)$ is paranormal. From the spectral mappin g theorem

We have $\sigma(h(T)) = h(\sigma(T) = \{h(0)\}$, so $h(0)I - h(T)$ quasi-nilpotent.

Since $h(0)I - h(T)$ is paranormal, then $h(0)I - h(T) = 0$,

and hence there are some $n \in \mathbb{N}$ and $\mu \in \mathbb{C}$, Such that

$$0 = h(0)I - h(T)$$
$$= \mu T^m \prod_{i=1}^{n}(\lambda_i I - T) \text{ with } \lambda_i \neq 0$$

Since all $\lambda_i I - T$ are invertible it then follows that $T^m = 0$.

We show now that for every isolated point λ of $\sigma(T)$

We have $p(\lambda I - T) = q(\lambda I - T) < \infty$,

Therefore λ is a pole of the resolvent.

If $\lambda \in iso\sigma(T)$, let P denote the spectral projection associted with $\{\lambda\}$,

$$M := K(\lambda I - T) = ker\,P$$

And $$N := H_0(\lambda I - T) = P(X).$$

Then, by the classical spectral decomposition, (M, N) is a GKD for $\lambda I - T$.

Since $\lambda I - T|N$ is quasi-nilpotent and algebraically paranormal,

then $\lambda I - T|N$ is nilpotent, and hence $\lambda I - T$ is of Kato type.

The SVEP for T and T^* at λ

Then implies by theorem 1.3.18 and theorem 1.3.19 that

Both $p(\lambda I - T)$ and $q(\lambda I - T)$ are finite. Hence λ is pole of resolvent.

To show the equality

$p_{00}(T) = \pi_{00}(T)$ it suffices to prove that inclusion $\pi_{00}(T) \subseteq p_{00}(T)$.

If $\lambda \in \pi_{00}(T)$, then $0 < \alpha(\lambda I - T) < \infty$ and the equality $p(\lambda I - T) = q(\lambda I - T) < \infty$ entails by theorem 1.3.17 that

$$\beta(\lambda I - T) = \alpha(\lambda I - T) < \infty,$$

So $\lambda \in \sigma(T)\backslash\sigma_b(T) = p_{00}(T)$.

1.4 Remark
Suppose that for a linear operator T we have $\alpha(T) < \infty$.

Then $\alpha(T^n) < \infty$ for all $n \in \mathbb{N}$.

This may be eassily seen by an an inductive argument.

Suppose that $dim \quad ker\ T^m < \infty$. Since $T(ker\ T^{n+1}) \subseteq ker\ T^n$,

Then the restriction

$T_0 = T|ker\ T^{n+1} : ker\ T^{n+1} \to ker\ T^n$ has kernal equal to $ker\ T$

So the canonical mapping

$\hat{T} = kerT^{n+1}/ker\ T \to ker\ T^n$ is injective.
Therfore we have

$$dim \quad ker\ T^{n+1}/ker\ T \leq dim \quad kerT^n < \infty,$$

and since $dim \quad ker\ T < \infty$

we then conclude that $dim \quad ker\ T^{n+1} < \infty$.

1.5 Lemma
Suppose that $T \in L(X)$ and $N$is nilpotent Such that $TN = NT$.

Then $T \in \mathcal{P}_0(X)$ iff $T + N \in \mathcal{P}_0(X)$.

Proof
Suppose that $N^p = 0$.

Observe first that without any assumption on T
we have

$$ker\ T \subseteq ker\ (T+N)^p \qquad \text{.....(1)}$$
and
$$ker\ (T+N) = ker\ T^p \qquad \text{.....(2)}$$

The first inclusion in(1) is clear.

Since for $x \in ker\ T$

we have $(T+N)^p x = N^p x$
$= 0.$

To show the second inclusion in (2)

Observe that if $x \in ker\ (T+N)$,

then $T^p x = (-1)^p N^p x$
$= 0.$

Suppose now that $T \in \mathcal{P}_0(X)$, or equivalently

$$p_{00}(T) = \pi_{00}(T).$$

We show first that

$$\pi_{00}(T) = \pi_{00}(T+N).$$

Let $\lambda \in \pi_{00}(T)$. There is no harm if we suppose $\lambda = 0$.

Form $\sigma(T+N) = \sigma(T)$ we see that $0 \in iso \quad (T+N)$.

Since $0 \in \pi_{00}(T)$,

then $\alpha(T) > 0$ and hence by the first inclusion in (1)

we have $\alpha(T+N)^p > 0$ and this obviously $\alpha(T+N) > 0$.

To show that $\alpha(T+N) < \infty$,

Observe that $ker\,(T+N) \subseteq ker\, T^p$

$$\subseteq H_0(T).$$

The equality $p_{00}(T) = \pi_{00}(T)$ is equivalent to saying that

$H_0(\lambda I - T)$ is finite dimensional for all$\lambda \in \pi_{00}(T)$ by theorem 1.3.20,

and hence $H_0(T)$ is finite dimensional.

Therefore $\alpha(T+N) < \infty$,

So $0 \in \pi_{00}(T+N)$

And the inclusion $\pi_{00}(T) \subseteq \pi_{00}(T+N)$ is proved.

To show the opposite inclusion,

assume that $0 \in \pi_{00}(T+N)$.

Clearly, $0 \in iso\sigma(T) = iso\sigma(T+N)$.

By assumption $\alpha(T+N) > 0$

So the second inclusion in (2) entails that $\alpha(T^p) > 0$

and this trivially implies that $\alpha(T) > 0$.

We also have $\alpha(T+N) < \infty$

and hence, by Remark 4.2.4 that $\alpha(T+N)^p < \infty$.

From the first inclusion in (1) we then conclude that $\alpha(T) < \infty$.

This show that $0 \in \pi_{00}(T)$, so the equality $\pi_{00}(T) = \pi_{00}(T+N)$ is proved.

Finally, if $T \in \mathcal{P}_0(X)$,

then
$$\begin{aligned} p_{00}(T+N) &= p_{00}(T) \\ &= \pi_{00}(T) \\ &= \pi_{00}(T+N), \end{aligned}$$

So $T + N \in \mathcal{P}_0(X)$.
Conversely, if $T + N \in \mathcal{P}_0(X)$ by symmetry

we have $p_{00}(T) = p_{00}(T + N)$
$= \pi_{00}(T + N)$
$= \pi_{00}((T + N) - N)$
$= \pi_{00}(T)$,

Hence $p_{00}(T) = \pi_{00}(T)$.

Therefore $T \in \mathcal{P}_0(X)$.

1.6 Lemma
Let $T \in L(X)$, X a Banach spce. If R is a Riesz operator that commutes with T,then

$$\pi_{0f}(T + R) \cap \sigma(T) \subseteq iso\sigma(T) \quad .....(1)$$

If $T \in L(H)$ is paranormal and N is a nilpotent operator commuting with T,

then $$\pi_{0f}(T + N) \cap \sigma(T) \subseteq p_{00}(T) \quad .....(2)$$

Proof
The inclusion (1) has been proved in lemma 1.3.16
Assume that $T$is paranormal, $N$is commuting with $T$and let $p \in \mathbb{N}$ be such that

$$N^p = 0.$$

Let $\lambda \in \pi_{0f}(T + N) \cap \sigma(T)$. by Remark 4.2.4

We have $\alpha((\lambda I - (T + N))^p) < \infty$. From the first inclusion of

$$ker\ T \subseteq ker\ (T + N)^p$$

and $$ker\ (T + N) \subseteq ker\ T^p.$$

it then follows that $\alpha(\lambda I - T) \leq \alpha((\lambda I - (T + N))^p) < \infty$.

Now, $\lambda \in iso\sigma(T + N) = iso\sigma(T)$ and T is paranormal, So by theorem 4.2.3

λ is a pole of the resolvent, or equivalently
$0 < p(\lambda I - T) = q(\lambda I - T) < \infty$. By theorem 1.3.17

We then conclude that $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$.

And hence $\lambda \in p_{00}(T)$, as required.

1.7 Theorem
Suppose that $T \in L(H)$ is paranormal, K algebraic and $TK = KT$.

Then Weyl's theorem holds for $T + K$.

Proof

Let $\sigma(K) = \{\mu_1, \mu_2 \ldots\ldots\ldots\ldots\ldots \mu_n\}$.

Denote by P_i the spectral projection associated with K and the spectral set $\{\mu_i\}$.

Let $Y_i := P_i(H)$ and $Z_i := ker\ P_i$. Then $H = Y_i \oplus Z_i$, the closed subspace Y_i and Z_i are invariant under T and K, and $\sigma(K|Y_i) = \{\mu_i\}$.

Define $K_i := K|Y_i$

and $T_i = T|Y_i$.

Clearly, the restriction T_i and K_i commute for every $i = 1,2,\ldots\ldots\ldots n$ and

$$\sigma(T+K) = \sigma(T_i + K_i) \cup \sigma((T+K)|Z_i) \qquad \ldots(1)$$

We show first that $T_i + K_i \in \mathcal{P}_0(Y_i)$ for every $i = 1,2,\ldots.,n$.

Let h be apolynomial such that

$$h(K) = 0.$$

Then $h(K_i) = h(K)|Y_i$
$= 0,$

and from
$$\{0\} = \sigma(h(K_i)) = h(\sigma(K_i)) = h(\{\mu_i\})$$

we obtain that $h(\mu_i) = 0$.

Write $h(\mu) = (\mu_i - \mu)^v q(\mu)$ with $q(\mu_i) \neq 0$.

Then $0 = h(K_i)$

$= (\mu_i I - K_i)^v q(K_i)$ where $q(K_i)$ is invertible.

Hence $(\mu_i I - K_i)^v = 0,$

So the operators $N_i := \mu_i I - K_i$ are nilpotent for all $i = 1,2,\ldots\ldots\ldots\ldots n$.

Note that
$$T_i + K_i = (\mu_i I + T_i) + (K_i - \mu_i I) = \mu_i I + T_i - N_i. \qquad \ldots(2)$$

Since every T_i is paranormal and hence algebraically paranormal,

then $\mu_i I + T_i$ is algebraically paranormal, and hence by theorem 4.2.3

$\mu_i I + T_i \in \mathcal{P}_0(Y_i)$. It follows from lemma 4.2.5 that

$$T_i + K_i = (\mu_i I + T_i) - N_i \in \mathcal{P}_0(Y_i).$$

We show now that $T + K \in \mathcal{P}_0(H)$.

Assume that $\lambda_o \in \pi_{00}(T+K)$.

Then $\lambda_o \in iso\sigma(T+K)$ and $\alpha(\lambda_0 I - (T+K)) < \infty$.

From (2) we have

$$(\lambda_0 - \mu_i)I - (T_i - N_i) = \lambda_0 I - (T_i + K_i) \qquad \ldots(3)$$

Fix $i \in \mathbb{N}$ such that $1 \leq i \leq n$

We consider two cases:
Case (i): $(\lambda_0 - \mu_i)I - T_i$ is invertible.

$N_i = \mu_i I - K_i$ being quasi-nilpotent and commuting with $(\lambda_0 - \mu_i)I - T_i$, from the equivalence

T invertible $\Leftrightarrow$ $T + Q$ invertible.

We deduce that

$$(\lambda_0 - \mu_i)I - T_i + N_i = \lambda_0 I - (T_i + K_i) \text{ is invertible}$$

and hence

$$H_0(\lambda_0 I - (T_i + K_i)) = ker(\lambda_0 - (T_i + K_i)) = \{0\} \qquad \text{.....(4)}$$

Case (ii): $(\lambda_0 - \mu_i)I - T_i$ is not invertible.

We have $\lambda_0 - \mu_i \in \sigma(T_i)$.

We claim that $\lambda_0 \in \pi_{00}(T_i + K_i)$

From (1) taking into account equality (2) We see that

$$\lambda_o \in iso\sigma(T_i + K_i) = iso\sigma(\mu_i I + (T_i - N_i)) \qquad \text{.....(5)}$$

From which we obtain $\lambda_0 - \mu_i \in iso\sigma(T_i - N_i)$.

Moreover, Since $\alpha(\lambda_0 I - (T + K)) < \infty$.

The inclusion $ker(\lambda_0 I - (T_i + K_i)) \subseteq ker(\lambda_0 I - (T + K))$

implies that

$$\alpha(\lambda_0 I - (T_i + K_i)) = \alpha((\lambda_0 - \mu_i)I - (T_i - N_i)) < \infty.$$

Therefore $\lambda_0 - \mu_i \in \pi_{0f}(T_i - N_i) \cap \sigma(T_i)$.

Since T_i is paranormal and T_i satisfies Weyl's theorem by lemma 4.2.6 and theorem 4.2.1 we then obtain that

$$\lambda_0 - \mu_i \in p_{00}(T_i) = \pi_{00}(T_i) = \sigma(T_i) \setminus \sigma_W(T_i)$$

From equalities T invertible $\Leftrightarrow$ $T + Q$ invertible.

We also have

$$\sigma(T_i) \setminus \sigma_W(T_i) = \sigma(T_i - N_i) \setminus \sigma_W(T_i - N_i),$$

So $(\lambda_0 - \mu_i)I - (T_i - N_i) = \lambda_0 I - (T_i + K_i)$ is Weyl.

On the other hand,

$$\lambda_0 - \mu_i \in \sigma(T_i) = \sigma(T_i - N_i),$$

Hence $(\lambda_0 - \mu_i)I - (T_i - N_i) = \lambda_0 I - (T_i + K_i)$ is not invertible.

This implies that $\alpha(\lambda_0 I - (T_i + K_i)) > 0$.

Other wise $\lambda_0 I - (T_i + K_i)$ being Weyl

we would have $\alpha(\lambda_0 I - (T_i + K_i)) = \beta(\lambda_0 I - (T_i + K_i)) = 0$.

And hence $\lambda_o \notin T_i + K_i)$,

which is contradicting (5)

Therefore $\lambda_o \in \pi_{00}(T_i + K_i)$.

Since $T_i + K_i \in \mathcal{P}_0(Y_i)$we then conclude that there exists $v_i \in \mathbb{N}$ such that

$$H_0(\lambda_0 I - (T_i + K_i)) = ker(\lambda_0 I - (T + K))^{v_i} \qquad \text{.....(6)}$$

From the equality $H_0(\lambda_0 I - (T+K)) = \oplus_{i=1}^{n} \quad H_0(\lambda_0 I - (T_i + K_i))$,
And taking into account equalities (4) and (6), we then conclude that

$$H_0(\lambda_0 I - (T+K)) = \oplus_{i=1}^{n} \quad ker(\lambda_0 I - (T_i + K_i))^{v_i}$$
$$= ker(\lambda_0 I - (T+K))^{v},$$

where $v := max\{v_1, v_2, \ldots\ldots\ldots\ldots\ldots, v_n\}$.

The last equalities hold for every $\lambda_0 \in \pi_{00}(T+K)$, so $T+K \in \mathcal{P}_0(H)$.

We show that $T+K$ has SVEP.

Clearly, every T_i has SVEP, Since T_i is paranormal. It follows from theorem 1.3.22 that

$T_i - N_i$ have SVEP,

and hence every $T_i + K_i = \mu_i I + (T_i - N_i)$ has SVEP By theorem 1.3.21
We obtain that

$T + K = \oplus_{i=1}^{n} \quad (T_i + K_i)$has SVEP. By theorem 4.2.2

We then conclude that $T+K$ satisfies Weyl's theorem, as desired.
It is well know that if for an operator $F \in L(X)$ there exists a positive integer n for which F^n is finite-dimensional, then F is algebraic.

1.8 Corollary
If $T \in L(H)$ is paranormal and $F$is an operator that commutes with$T$ and such that $F^n$is finite dimensional operator for some positive integer$n$, then $T+F$ satisfies Weyl's theorem.

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