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Role of logarithmic and exponential functions in an infinite product

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Abstract
Logarithmic and exponential functions play an important role in complex analysis. There are many areas in complex analysis where logarithmic and exponential functions are used. Infinite product is one of them. In this research paper, we will discuss important results of Infinite products with logarithmic and exponential functions. Exponential and logarithmic functions are very important when we discuss them in an infinite product, because they give us many useful results with infinite product. We will discuss a very important relation of logarithmic and exponential function in an infinite product.

Keywords: infinite, partial, converges

Introduction

An expression of the type $(1+a_1)(1+a_2)(1+a_3)\ldots\ldots$ (1.1)
(Where all a_i's are not equals to 1) is called an infinite product.
We will denote n^{th} partial product by $P_n = (1+a_1)(1+a_2)(1+a_3)\ldots\ldots(1+a_n)$.
If we can find a value k which is non zero such that $\lim_{n\to\infty} P_n = k$, then we say infinite product (1.1)
converges to k. The necessary condition for the product $(1+a_1)(1+a_2)(1+a_3)\ldots\ldots$
to converge is that $a_n \to 0$, however this condition is not sufficient. There is one more powerful result:-
An infinite product (1.1) converges iff the sum
$\log(1+a_1)+\log(1+a_2)+\log(1+a_3)+\log(1+a_4)+\ldots\ldots\ldots\ldots$
converges. Therefore there are many important results in infinite product spaces. We will also discuss a very important relation of logarithmic and exponential function in an infinite product.

Method

Let $a_1, a_2, \ldots a_N$ be complex numbers. We let $p_N = (1+a_1)(1+a_2)\ldots(1+a_N)$ and $q_N = (1+|a_1|)(1+|a_2|)\ldots(1+|a_N|)$,
then $\log(1+|p_N-1|) \le \log(1+|a_1|)+\log(1+|a_2|)\ldots+\log(1+|a_N|) \le |a_1|+|a_2|+\ldots|a_N|$.
Proof: For $x\ge0$, we have $e^x = 1+x+x^2/2!+\ldots \ge 1+x$ as $x\ge0$
Therefore $1+x \le e^x$ for $x\ge0$ (1.2)
Put $x=|a_1|, |a_2|, \ldots\ldots, |a_N|$ in (1.2), we get
$1+|a_1| \le \exp|a_1|$, $1+|a_2| \le \exp|a_2|$, …………, $1+|a_N| \le \exp|a_N|$
Multiplying these equations,
$(1+|a_1|)(1+|a_2|)\ldots\ldots, (1+|a_N|) \le \exp|a_1| \exp|a_2| \ldots\ldots \exp|a_N|$
$\Rightarrow q_N \le \exp(|a_1|+|a_2|+\ldots\ldots+|a_N|)$
$\Rightarrow q_N \le \exp \sum_1^N |a_j|$ (1.3)
Now we prove $|p_N-1| \le q_N-1$ (1.4)
We will use mathematical induction for proving this.
Since $|1+a_1-1| \le 1+|a_1|-1$ therefore $|p_1-1| \le q_1-1$ and hence the result is true for n=1.
We assume that result is true for n=k therefore,
$|p_k-1| \le q_k-1$ (1.5)

Now for n=k+1,
$p_{k+1}-1 = (1+a_1)(1+a_2)\ldots\ldots(1+a_k)(1+a_{k+1})-1$
$\Rightarrow p_{k+1}-1 = [(1+a_1)(1+a_2)\ldots\ldots(1+a_k)](1+a_{k+1})-1$
$\Rightarrow p_{k+1}-1 = p_k(1+a_{k+1})-1$
$\Rightarrow p_{k+1}-1 = p_k(1+a_{k+1})-1-a_{k+1}+a_{k+1}$ by adding and subtracting a_{k+1}
$\Rightarrow p_{k+1}-1 = [p_k(1+a_{k+1})-1(1+a_{k+1})] + a_{k+1}$
$\Rightarrow p_{k+1}-1 = (p_k-1)(1+a_{k+1}) + a_{k+1}$
$\Rightarrow |p_{k+1}-1| \le |p_k-1|(1+|a_{k+1}|) + |a_{k+1}|$
$\le (q_k-1)(1+|a_{k+1}|) + |a_{k+1}|$ by using (1.5)
$= [q_k(1+|a_{k+1}|)-1(1+|a_{k+1}|)] + |a_{k+1}|$
$= q_k(1+|a_{k+1}|)-1-|a_{k+1}| + |a_{k+1}|$
$= q_k(1+|a_{k+1}|)-1$

Therefore we have $|p_{k+1}-1| \le q_k(1+|a_{k+1}|)-1$ which implies
Since $q_N = (1+|a_1|)(1+|a_2|)\ldots(1+|a_N|)$ therefore,
$|p_{k+1}-1| \le (1+|a_1|)(1+|a_2|)\ldots(1+|a_k|)(1+|a_{k+1}|)-1$
$\Rightarrow |p_{k+1}-1| \le q_{k+1}-1$
Hence the inequality $|p_N-1| \le q_N-1$ is true by mathematical induction.
We can write it as $1+|p_N-1| \le q_N$ (1.6)
Combining results (1.3) and (1.6), we get,
$1+|p_N-1| \le q_N \le \exp \sum_1^N |a_j|$
Taking log on both sides, we get
$\log(1+|p_N-1|) \le \log q_N \le \log(\exp \sum_1^N |a_j|)$ which gives,
$\log(1+|p_N-1|) \le \log[(1+|a_1|)(1+|a_2|)\ldots(1+|a_N|)] \le |a_1|+|a_2|+\ldots|a_N|$.
$\log(1+|p_N-1|) \le \log(1+|a_1|)+\log(1+|a_2|)\ldots+\log(1+|a_N|) \le |a_1|+|a_2|+\ldots+|a_N|$.
This is our desired result.

Results and Discussion

Infinite product is very important when discussed with logarithm and exponential function. There are many results of this topic. Here we have also discussed a result. If we let p_N and q_N as special types of infinite product i.e.
$p_N = (1+a_1)(1+a_2)\ldots(1+a_N)$ and
$q_N = (1+|a_1|)(1+|a_2|)\ldots(1+|a_N|)$,

Then a very important result can be established i.e.
log $(1+|p_N-1|) \leq$ log $(1+|a_1|)$+log$(1+|a_2|)\ldots$+log$(1+|a_N|) \leq |a_1|+|a_2|+\ldots+|a_N|$. Where a_1, a_2,….a_N be complex numbers

Conclusion

Exponential and logarithmic functions do not only play important role in basic sciences, but they also play very important role in complex analysis also. Infinite product is one of them. Many results can be established with logarithmic and exponential functions when taken with infinite products.

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