International Journal of Advanced Research and Development
ISSN: 2455-4030
Impact Factor: RJIF 5.24
www.advancedjournal.com
Volume 2; Issue 6; November 2017; Page No. 196-199

Inclusion properties for some subclasses of analytic multivalent functions involving generalized integral operator

Manita Bhagtani
Department of Mathematics, SS Jain Subodh PG Autonomous College, Jaipur, Rajasthan, India

Abstract
In the present paper some new subclasses of analytic multivalent functions are introduced which are defined by certain generalized integral operator. We shall establish inclusion relations for these subclasses.

Keywords: analytic functions, multivalent functions, starlike functions, convex functions, logarithmic differentiation, integral operator

1. Introduction

Let A_p denote the class of functions f analytic and p-valent in the open unit disk U = {z : | z | < 1 } and be given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in \mathbf{N}) \qquad \ldots (1.1)$$

A function $f \in A_p$ is called p-valent *starlike function* of order γ $(0 \le \gamma < p)$, if

$$\mathrm{Re}\left\{ \frac{z f'(z)}{f(z)} \right\} > \gamma \qquad (z \in U) \qquad \ldots (1.2)$$

We denote this class by $S_p^*(\gamma)$. On the other hand a function of $\in A_p$ is called p-valent *convex function* of order γ $(0 \le \gamma < p)$, if

$$\mathrm{Re}\left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \gamma \qquad (z \in U) \qquad \ldots (1.3)$$

2000 Mathematics Subject Classification. 30 C 45, 26 a 33.
We shall denote this class by $C_p(\gamma)$. Further it is observed that

$$f \in C_p(\gamma) \text{ iff } z f' \in S_p^*(\gamma) \qquad (P \in \mathbf{N}, z \in U)$$

Further suppose that

$$h_p[(\alpha_q);(\beta_r);z] = z^p \, {}_qF_r(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_r;z)$$

$$= z^p + \sum_{n=p+1}^{\infty} B_p^{(\alpha_q),(\beta_r)}(n)\, z^n \qquad \ldots (1.4)$$

$(q \le r+1; \alpha_i \in R; \beta_j \in R \setminus Z_0^-; Z_0^- = \{0,-1,-2,\ldots\};$
i = 1… q; j = 1… r; $z \in U$)

Where ${}_qF_r$ is the generalized hypergeometric function and

$$B_p^{(\alpha_q),(\beta_r)}(n) = \frac{(\alpha_1)_{n-p}(\alpha_2)_{n-p}\dots(\alpha_q)_{n-p}}{(\beta_1)_{n-p}(\beta_2)_{n-p}\dots(\beta_r)_{n-p}(n-p)!}. \quad \dots (1.5)$$

Corresponding to the function $h_p[(\alpha_q);(\beta_r);z]$, Dziok and Srivastava [2, p.3, Eq. (3)] introduced a linear operator H_{p,α_q,β_r} defined by the convolution

$$H_{P,\alpha_q,\beta_r} f(z) = h_p[(\alpha_q);(\beta_r);z] * f(z) \quad \dots (1.6)$$

Or equivalently by

$$H_{P,\alpha_q,\beta_r} f(z) = z^p + \sum_{n=p+1}^{\infty} B_p^{(\alpha_q),(\beta_r)}(n)\, a_n z^n \quad (z \in U). \quad \dots (1.7)$$

Here $*$ stands for the convolution of two analytic multivalent functions f and g of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,\ g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n\ (a_n, b_n \geq 0,\ p \in N)$$

And is defined by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n. \quad \dots (1.8)$$

The linear operator $H_{P,\alpha_q,\beta_r} f(z)$ includes various other linear operators considered earlier by Hohlov [4], Carlson-Shaffer [1], Goyal and Bhagtani [3], Ruscheweyh [6] etc.

Next by using the operator H_{P,α_q,β_r} we introduce the following classes of analytic functions for $\phi \in H$; $f \in A_p$; $\alpha_q > -1$ and $\beta_r \geq 1$

$$S^*_{p,\alpha_q,\beta_r}(\phi) = \{f : f \in A_p \text{ and } H_{p,\alpha_q,\beta_r} f(z) \in S_p^*(\phi)\}$$

$$C_{p,\alpha_q,\beta_r}(\phi) = \{f : f \in A_p \text{ and } H_{p,\alpha_q,\beta_r} f(z) \in C_p(\phi)\}$$

We also note that

$$f(z) \in C_{p,\alpha_q,\beta_r}(\phi) \Leftrightarrow z f'(z) \in S^*_{p,\alpha_q,\beta_r}(\phi) \quad \dots (1.9)$$

The results contained in the following lemma due to Miller and Mocanu [5] will be required to establish our main results:

Lemma

(Miller and Mocanu [5]). *Let $w(u,v)$ be a complex valued function, $w : D \to C$, $D \subset C \times C$ (C is a complex plane) and let $u = u_1 + i u_2$ and $v = v_1 + i v_2$. Suppose that the function $w(u,v)$ satisfies the following conditions:*

(i) $w(u,v)$ is continuous in D,

(ii) $(1, 0) \in D$ and $Re\{w(1,0)\} > 0$,

(iii) $Re\{w(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1+u_2^2)/2$.

Let $h(z) = 1 + c_1 z + c_2 z^2 + \dots$ be analytic in U, such that $(h(z), zh'(z)) \in D$ for all $z \in U$. If $Re\{w(h(z), zh'(z))\} > 0$ $(z \in U)$, then $Re\, h(z) > 0$ for all $z \in U$.

2. Inclusion Relations

We establish the following inclusion Theorems:

Theorem 1. *Let* $\alpha_q > -1$ and $\beta_r \geq 1$. *Then*

$$S^*_{p,\alpha_q,\beta_r}(\phi) \subset S^*_{p,\alpha_q,\beta_r}(\alpha_1 + 1)(\phi)$$

Proof. Let $f(z) \in S^{*}_{p,\alpha_q,\beta_r}(\phi)$.

Setting $\frac{z(H_{P,\alpha_q,\beta_r} f(z))'}{H_{P,\alpha_q,\beta_r} f(z)} - \gamma = (p-\gamma)\, h(z),$

Where h (z) = 1 + $c_1 z + c_2 z^2 + \ldots$ and using the identity

$$z(H_{P,\alpha_{q,=1}\beta_r} f(z))' = (\alpha_1) H_{P,\alpha_q,\beta_r}(\alpha_1+1) f(z) - (\alpha_1 - p) H_{P,\alpha_q,\beta_r} f(z), \qquad \ldots (2.1)$$

$(\alpha_q > -1 \text{ and } \beta_r \geq 1 \text{ and } f \in A_p)$

Where

$$H_{P,\alpha_q,\beta_r}(\alpha_1+1) f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1+1)_{n-p}(\alpha_2)_{n-p}\ldots(\alpha_q)_{n-p}}{(\beta_1)_{n-p}(\beta_2)_{n-p}\ldots(\beta_r)_{n-p}(n-p)!} a_n z^n$$

We find that

$$\frac{H_{P,\alpha_q,\beta_r}(\alpha_1+1) f(z)}{H_{P,\alpha_q,\beta_r} f(z)} = \frac{1}{(\alpha_1)}\left[\frac{z(H_{P,\alpha_q,\beta_r} f(z))'}{H_{P,\alpha_q,\beta_r} f(z)} + \alpha_1 - p\right]$$

$$= \frac{1}{(\alpha_1)}\{\gamma + (p-\gamma) h(z) + \alpha_1 - p\}. \qquad \ldots (2.2)$$

Next using the logarithmic differentiation on both sides of (2.2), we obtain

$$\frac{z(H_{P,\alpha_q,\beta_r}(\alpha_1+1) f(z))'}{H_{P,\alpha_q,\beta_r}(\alpha_1+1) f(z)} = \frac{z(H_{P,\alpha_q,\beta_r} f(z))'}{H_{P,\alpha_q,\beta_r} f(z)} + \frac{(p-\gamma) z h'(z)}{(p-\gamma) h(z) + \gamma + \alpha_1 - p}$$

$$= \gamma + (p-\gamma) h(z) + \frac{(p-\gamma) z h'(z)}{(p-\gamma) h(z) + \gamma + \alpha_1 - p},$$

or

$$\frac{z(H_{P,\alpha_q,\beta_r}(\alpha_1+1) f(z))'}{H_{P,\alpha_q,\beta_r}(\alpha_1+1) f(z)} - \gamma = (p-\gamma) h(z) + \frac{(p-\gamma) z h'(z)}{(p-\gamma) h(z) + \gamma + \alpha_1 - p} \qquad \ldots. (2.3)$$

Now we form the function w (u,v) by taking u = h(z), v = z h' (z) in (2.3) as:

$$w(u,v) = (p-\gamma) u + \frac{(p-\gamma) v}{(p-\gamma) u + \alpha_1 - p}, \qquad \ldots (2.4)$$

Then we see that

(i) W (u,v) is continuous in $D = \left(C - \left\{\frac{\gamma + \alpha_1 - p}{\gamma - p}\right\}\right) \times C$,

(ii) (1, 0) $\in$ D and Re {w (1,0)} = p - γ > 0 ,

(iii) For all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1+u_2^2)/2,$

$$\operatorname{Re} w(iu_2, v_1) = \frac{(\gamma + \alpha_1 - p)(p - \gamma) v_1}{(\gamma + \alpha_1 - p)^2 + (p - \gamma)^2 u_2^2}$$

$$\leq -\frac{(\gamma + \alpha_1 - p)(p - \gamma)(1 + u_2^2)}{2[(\gamma + \alpha_1 - p)^2 + (p - \gamma)^2 u_2^2]} \leq 0.$$

This implies that the function w (u,v) satisfies the conditions of Lemma . This shows that if Re w (h(z), z h' (z)) > 0 (z $\in$ U), then Re h(z) > 0, (z $\in$ U).

Thus if f (z) $\in S^*_{p,\alpha_q,\beta_r}(\phi)$ then $f(z) \in S^*_{p,\alpha_q,\beta_r}(\alpha_1 + 1)(\phi)$.

Theorem 2. *Let* α_q > - 1 and β_r ≥ 1. *Then*

$$C^*_{p,\alpha_q,\beta_r}(\phi) \subset C^*_{p,\alpha_q,\beta_r}(\alpha_1 + 1)(\phi)$$

Proof.

$$f \in C_{p,\alpha_q,\beta_r}(\gamma) \Leftrightarrow H_{P,\alpha_q,\beta_r} f(z) \in C_p(\gamma)$$

$$\Leftrightarrow z(H_{P,\alpha_q,\beta_r} f)' \in S_p^*(\gamma) \Leftrightarrow H_{P,\alpha_q,\beta_r}(zf') \in S_p^*(\gamma)$$

$$\Leftrightarrow zf' \in S^*_{p,\alpha_q,\beta_r}(\gamma) \quad \Rightarrow zf' \in S^*_{p,\alpha_q,\beta_r}(\alpha_1 + 1)(\gamma)$$

$$\Leftrightarrow (H_{P,\alpha_q,\beta_r}(\alpha_1 + 1))(zf') \in S_p^*(\gamma) \Leftrightarrow z((H_{P,\alpha_q,\beta_r}(\alpha_1 + 1) f)' \in S_p^*(\gamma)$$

$$\Leftrightarrow H_{P,\alpha_q,\beta_r}(\alpha_1 + 1) f \in C_p(\gamma) \Leftrightarrow f \in C^*_{p,\alpha_q,\beta_r}(\alpha_1 + 1)(\gamma).$$

References

1. Carlson BC, Shaffer DB, Starlike, prestarlike hypergeometric functions, SIAM J. Math. Anal., 1984; 15:737-745.
2. Dziok J, Srivastava HM. Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput, 1999; 103:1-13.
3. Goyal SP, Bhagtani Manita. Applications of certain integral operators to a class of analytic multivalent functions, Journal of Rajasthan Academy of Physical Sciences, 2007; 6:223-236.
4. Hohlov, Yu E. Operators and operations in the class of univalent functions, Izv. Vyss. Ueebn Zaved. Matematika, 1978; 10:83-89.
5. Miller SS, Mocanu PT. Second-order differential inequalities in the complex-plane, J. Math. Anal. Appl., 1978; 65:289-305.
6. Ruscheweyh S. New criteria for univalent functions, Proc. Amer. Math. Soc., 1975; 49:109-115.