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New subclass of univalent function associated with a fractional calculus operators

[1] Pawan Chanchal, [2] AK Arora, [3] Jyoti Taneja
[1] Departmentof Mathematics, Government College Kekri, Ajmer, Rajasthan, India
[2] Department of Mathematics, Government College of Ajmer, Ajmer, Rajasthan, India
[3] Department of Mathematics, Bhag Singh Khalsa College for Women, Abohar, Punjab, India

Abstract
In this paper, by making use of generalized hypergeometric function we define a new class $X(a, b, c;\ \beta, \delta, k)$ of univalent functions in the open unit disk. We derived coefficient estimate, distortion theorems, extreme points and application to fractional calculus operator for the function belonging to this class.

Keywords: hadamard product, generalized hypergeometric functions, fractional integral operator, coefficient estimate, distortion theorem

1. Introduction

Let Ω be the class of analytic and univalent function in the open unit disk $U = \{ z \in C :\ |z| < 1\}$ is of the form

$$f(z) = z + \sum_{k=2}^{\infty} \alpha_k z^k \tag{1}$$

Let f ∈ Ω given by (1) and g ∈ Ω given by

$$g(z) = z + \sum_{k=2}^{\infty} \beta_k z^k \tag{2}$$

We define the convolution product (or Hadamard) of f and g by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} \alpha_k \beta_k z^k = (g * f)(z);\ (z \in \mathcal{U}), \tag{3}$$

Let Δ is subclass of Ω consisting of the functions of the form:

$$f(z) = z - \sum_{k=2}^{\infty} \alpha_k z^k \tag{4}$$

Definition 1:- The generalized hypergeometric function $_2R_1$ (a, b, c; k; z) [1] is defined by

$$ {}_2R_1\ (a, b, c;\ k;\ z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+kn) z^n}{\Gamma(c+kn)\ (n)!},\ \ k \in R, k > 0, |z| < 1, \tag{5}$$

Where Re(c-a-b)>0 and $(a)_n$ Pochhammer symbol is given by $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$.

Definition 2:- Let f(z) be in form (4), by using (3) we define a operator $\Phi(a, b, c; k)$: Δ→ Δ by

$$\mu(a, b, c; k) f(z) = \left(z\ \ {}_2R_1\ (a, b, c;\ k;\ z)\right) * f(z) = \left(z \left(\frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b+kn) z^n}{\Gamma(c+kn)\ (n)!} \right) \right) * \left(z - \sum_{n=2}^{\infty} \alpha_n z^n \right)$$

$$\mu(a, b, c; k) f(z) = z - \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_{n-1} \Gamma(b+k(n-1)) \alpha_n z^n}{\Gamma(c+k(n-1))\ (n-1)!}\ \ a, b, c \in N; z \in U, k > 0 \tag{6}$$

Definition 3:- Let f(z) be in the form (4) is said to be in subclass X (a, b, c; β, δ, k) if and only if

$$\left| \frac{z\left(\mu(a,b,c;k) f(z)\right)''' - \delta\left(\mu(a,b,c;k) f(z)\right)''}{\left(\mu(a,b,c;k) f(z)\right)'' + 2(1-\delta)} \right| < \beta,\ \ z \in U, 0 \ \leq \delta \leq 1, 0 < \beta \leq 1, \tag{7}$$

2. Coefficient Estimate

Theorem 1:- Let the function f(z) defined by (4). Then $f(z) \in X(\mathrm{a,b,c};\ \beta,\delta,\mathrm{k})$ if and only if

$$\sum_{n=2}^{\infty} n(n-1)(n-2+\delta+\beta)\frac{\Gamma(c)(a)_{n-1}\Gamma(b+k(n-1))\alpha_n}{\Gamma(b)\Gamma(c+k(n-1))\,(n-1)!} \leq 2\beta(1-\delta) \quad (8)$$

above result is sharp for the function of the form

$$f(z) = z - \frac{2\beta(1-\delta)}{n(n-1)(n-2+\delta+\beta)\frac{\Gamma(c)(a)_{n-1}\Gamma(b+k(n-1))}{\Gamma(b)\Gamma(c+k(n-1))\,(n-1)!}}\ z^n, n \geq 2 \quad (9)$$

Proof:- Suppose that the inequalities (8) holds true then we obtain

$$\left|z\left(\mu(a,b,c;k)f(z)\right)''' - \delta\left(\mu(a,b,c;k)f(z)\right)''\right| - \beta\left|\left(\mu(a,b,c;k)f(z)\right)'' + 2(1-\delta)\right|$$

$$= \left|-\sum_{n=2}^{\infty} n(n-1)(n-2)\frac{\Gamma(c)(a)_{n-1}\Gamma\left(b+k(n-1)\right)}{\Gamma(b)\Gamma\left(c+k(n-1)\right)(n-1)!}\alpha_n z^{n-2} + \delta\sum_{n=2}^{\infty} n(n-1)\frac{\Gamma(c)(a)_{n-1}\Gamma\left(b+k(n-1)\right)}{\Gamma(b)\Gamma\left(c+k(n-1)\right)(n-1)!}\alpha_n z^{n-2}\right|$$

$$-\beta\left|2(1-\delta) - \sum_{n=2}^{\infty} n(n-1)\frac{\Gamma(c)(a)_{n-1}\Gamma\left(b+k(n-1)\right)}{\Gamma(b)\Gamma\left(c+k(n-1)\right)(n-1)!}\alpha_n z^{n-2}\right|$$

$$\leq \sum_{n=2}^{\infty} n(n-1)(n-2+\delta+\beta)\frac{\Gamma(c)(a)_{n-1}\Gamma\left(b+k(n-1)\right)}{\Gamma(b)\Gamma\left(c+k(n-1)\right)(n-1)!}\alpha_n - 2\beta(1-\delta)$$

$\leq 0.$

Hence by maximum modulus principle, $f(z) \in X(\mathrm{a,b,c};\ \beta,\delta,\mathrm{k})$

Conversely:- Suppose $f(z) \in X(\mathrm{a,b,c};\ \beta,\delta,\mathrm{k})$ then

$$\left|\frac{z\left(\mu(a,b,c;k)f(z)\right)''' - \delta\left(\mu(a,b,c;k)f(z)\right)''}{\left(\mu(a,b,c;k)f(z)\right)'' + 2(1-\delta)}\right| < \beta, z \in U$$

then

$$\left|z\left(\mu(a,b,c;k)f(z)\right)''' - \delta\left(\mu(a,b,c;k)f(z)\right)''\right| < \beta\left|\left(\mu(a,b,c;k)f(z)\right)'' + 2(1-\delta)\right|$$

we get

$$\left|-\sum_{n=2}^{\infty} n(n-1)(n-2)\frac{\Gamma(c)(a)_{n-1}\Gamma\left(b+k(n-1)\right)}{\Gamma(b)\Gamma\left(c+k(n-1)\right)(n-1)!}\alpha_n z^{n-2} + \delta\sum_{n=2}^{\infty} n(n-1)\frac{\Gamma(c)(a)_{n-1}\Gamma\left(b+k(n-1)\right)}{\Gamma(b)\Gamma\left(c+k(n-1)\right)(n-1)!}\alpha_n z^{n-2}\right|$$

$$< \beta\left|2(1-\delta) - \sum_{n=2}^{\infty} n(n-1)\frac{\Gamma(c)(a)_{n-1}\Gamma\left(b+k(n-1)\right)}{\Gamma(b)\Gamma\left(c+k(n-1)\right)(n-1)!}\alpha_n z^{n-2}\right|$$

thus

$$\sum_{n=2}^{\infty} n(n-1)(n-2+\delta+\beta)\frac{\Gamma(c)(a)_{n-1}\Gamma(b+k(n-1))\alpha_n}{\Gamma(b)\Gamma(c+k(n-1))\,(n-1)!} \leq 2\beta(1-\delta)$$

finally we can see sharpness follows for the function defined in (9).

3. Distortion Bounds

Theorem 2:- Let the function f (z) defined by (4) be in the class $X(\mathrm{a,b,c};\ \beta,\delta,\mathrm{k})$ then we have

$$|z| - \frac{\beta(1-\delta)}{(\delta+\beta)}.\frac{\Gamma(b)\Gamma(c+k)}{a\,\Gamma(c)\Gamma(b+k)}\ |z|^2 \leq |f(z)| \leq |z| + \frac{\beta(1-\delta)}{(\delta+\beta)}.\frac{\Gamma(b)\Gamma(c+k)}{a\,\Gamma(c)\Gamma(b+k)}\ |z|^2, \quad (10)$$

Furthermore

$$1-\frac{2\beta(1-\delta)}{(\delta+\beta)}.\frac{\Gamma(b)\Gamma(c+k)}{a\,\Gamma(c)\Gamma(b+k)}\,|z| \le |f'(z)| \le 1+\frac{2\beta(1-\delta)}{(\delta+\beta)}.\frac{\Gamma(b)\Gamma(c+k)}{a\,\Gamma(c)\Gamma(b+k)}\,|z|, \tag{11}$$

the result is sharp for the function defined by

$$f(z)=z-\frac{\beta(1-\delta)}{(\delta+\beta)}.\frac{\Gamma(b)\Gamma(c+k)}{a\,\Gamma(c)\Gamma(b+k)}\,z^2 \tag{12}$$

Proof :- It is easy to see from theorem1 that

$$2(\delta+\beta)\frac{a\,\Gamma(c)\Gamma(b+k)}{\Gamma(b)\Gamma(c+k)}\sum_{n=2}^{\infty}\alpha_n \le \sum_{n=2}^{\infty} n(n-1)(n-2+\delta+\beta)\frac{\Gamma(c)(a)_{n-1}\Gamma(b+k(n-1))\alpha_n}{\Gamma(b)\Gamma(c+k(n-1))\,(n-1)!} \le 2\beta(1-\delta)$$

Then

$$\sum_{n=2}^{\infty}\alpha_n \le \frac{\beta(1-\delta)}{(\delta+\beta)}.\frac{\Gamma(b)\Gamma(c+k)}{a\,\Gamma(c)\Gamma(b+k)} \tag{13}$$

by using (13), we have

$$|f(z)| \ge |z|-|z|^2\sum_{n=2}^{\infty}\alpha_n \ge |z|-\frac{\beta(1-\delta)}{(\delta+\beta)}.\frac{\Gamma(b)\Gamma(c+k)}{a\,\Gamma(c)\Gamma(b+k)}\,|z|^2,$$

and

$$|f(z)| \le |z|+|z|^2\sum_{n=2}^{\infty}\alpha_n \le |z|+\frac{\beta(1-\delta)}{(\delta+\beta)}.\frac{\Gamma(b)\Gamma(c+k)}{a\,\Gamma(c)\Gamma(b+k)}\,|z|^2,$$

Which proves the assertion (10)
from (13) and theorem1, it follows also that

$$\sum_{n=2}^{\infty} n\alpha_n \le \frac{2\beta(1-\delta)}{(\delta+\beta)}.\frac{\Gamma(b)\Gamma(c+k)}{a\,\Gamma(c)\Gamma(b+k)}$$

consequently, we have

$$|f'(z)| \ge 1-|z|\sum_{n=2}^{\infty} n\alpha_n \ge 1-\frac{2\beta(1-\delta)}{(\delta+\beta)}.\frac{\Gamma(b)\Gamma(c+k)}{a\,\Gamma(c)\Gamma(b+k)}\,|z|,$$

and

$$|f'(z)| \le 1+|z|\sum_{n=2}^{\infty} n\alpha_n \le 1+\frac{2\beta(1-\delta)}{(\delta+\beta)}.\frac{\Gamma(b)\Gamma(c+k)}{a\,\Gamma(c)\Gamma(b+k)}\,|z|,$$

This proves the assertion (11). Since each of equalities in (10) and (11) is satisfied by the function f(z) given by (12), which proves our theorem.

4. Extreme Points

Theorem 3:- Let $f_1(z)=z$ and $f_n(z)=z-\frac{2\beta(1-\delta)}{n(n-1)(n-2+\delta+\beta)\frac{\Gamma(c)(a)_{n-1}\Gamma(b+k(n-1))}{\Gamma(b)\Gamma(c+k(n-1))\,(n-1)!}}\,z^n$, n=2,3,.. Then $f(z)\in X(a,b,c;\ \beta,\delta,k)$ iff it can be expressed in the form $f(z)=\sum_{n=1}^{\infty} b_n f_n(z)$ Where $b_n \ge 0$ and $\sum_{n=1}^{\infty} b_n = 1$.

Proof:- Firstly, let

$$f(z)=\sum_{n=1}^{\infty} b_n f_n(z) = b_1 z+\sum_{n=2}^{\infty} b_n\left[z-\frac{2\beta(1-\delta)}{n(n-1)(n-2+\delta+\beta)\frac{\Gamma(c)(a)_{n-1}\Gamma(b+k(n-1))}{\Gamma(b)\Gamma(c+k(n-1))\,(n-1)!}}\,z^n\right]$$

$$=z\left(b_1+\sum_{n=2}^{\infty} b_n\right)-\sum_{n=2}^{\infty}\frac{2\beta(1-\delta)}{n(n-1)(n-2+\delta+\beta)\frac{\Gamma(c)(a)_{n-1}\Gamma(b+k(n-1))}{\Gamma(b)\Gamma(c+k(n-1))\,(n-1)!}}\,z^n$$

$$=z-\sum_{n=2}^{\infty}\frac{2\beta(1-\delta)}{n(n-1)(n-2+\delta+\beta)\frac{\Gamma(c)(a)_{n-1}\Gamma(b+k(n-1))}{\Gamma(b)\Gamma(c+k(n-1))\,(n-1)!}}\,z^n$$

Therefore, $(z) \in X(\mathrm{a,b,c};\ \beta,\delta,\mathrm{k})$, since

$$\sum_{n=2}^{\infty}\frac{2\beta(1-\delta)}{n(n-1)(n-2+\delta+\beta)\frac{\Gamma(c)(a)_{n-1}\Gamma(b+k(n-1))}{\Gamma(b)\Gamma(c+k(n-1))\,(n-1)!}}\cdot\frac{n(n-1)(n-2+\delta+\beta)\frac{\Gamma(c)(a)_{n-1}\Gamma(b+k(n-1))}{\Gamma(b)\Gamma(c+k(n-1))\,(n-1)!}}{2\beta(1-\delta)}=\sum_{n=2}^{\infty}b_n=1-b_1<1$$

Conversely:- Assume that $f(z) \in X(\mathrm{a,b,c};\ \beta,\delta,\mathrm{k})$, then by (8) we may set

$$b_n=\frac{n(n-1)(n-2+\delta+\beta)\frac{\Gamma(c)(a)_{n-1}\Gamma(b+k(n-1))}{\Gamma(b)\Gamma(c+k(n-1))\,(n-1)!}}{2\beta(1-\delta)}\,\alpha_n, n\geq 2\ and\ 1-\sum_{n=2}^{\infty}b_n=b_1$$

Thus

$$f(z)=z-\sum_{n=2}^{\infty}\alpha_n z^n=z-\sum_{n=2}^{\infty}\frac{2\beta(1-\delta)\mathrm{b_n}}{n(n-1)(n-2+\delta+\beta)\frac{\Gamma(c)(a)_{n-1}\Gamma(b+k(n-1))}{\Gamma(b)\Gamma(c+k(n-1))\,(n-1)!}}z^n$$

$$=z-\sum_{n=2}^{\infty}b_n\big(z-f_n(z)\big)=z\left(1-\sum_{n=2}^{\infty}b_n\right)+\sum_{n=2}^{\infty}b_nf_n(z)$$

$$=b_1z+\sum_{n=2}^{\infty}b_nf_n(z)=\sum_{n=1}^{\infty}b_nf_n(z)$$

This proves the theorem.

5. Application to the Fractional Calculus

Many definitions of fractional calculus are given in the literature ([2, 3, 5]) i.e. fractional derivatives and fractional integrals. We here recall the following definition which is used by Owa [6] and by Srivastava and Owa [4].

Definition 4:- The fractional integral of order μ is defined for a function f (z), by

$$D_z^{-\mu}f(z)=\frac{1}{\Gamma(\mu)}\int_0^z\frac{f(t)}{(z-t)^{1-\mu}}\,dt\ \ (\mu>0),$$

Where f(z) is an analytic function in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-t)^{\mu-1}$ is removed by requiring log(z-t) to the real when z-t > 0.

Definition 5:- The fractional derivative of order μ is defined, for a function f(z), by

$$D_z^{\mu}f(z)=\frac{1}{\Gamma(1-\mu)}\frac{d}{dz}\int_0^z\frac{f(t)}{(z-t)^{\mu}}\,dt\ \ (0\leq\mu<1),$$

Where f(z) is an analytic function in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-t)^{-\mu}$ is removed by requiring log(z-t) to the real when z-t > 0.

Definition6:- under the hypotheses of definition5, the fractional derivative of order n+μ is defined by

$$D_z^{n+\mu}f(z)=\frac{d^n}{dz^n}D_z^{\mu}f(z)\ (0\leq\mu<1;n\in\mathbb{N}_0)$$

Now, we state the definition of fractional integral operator given by [7].

Definition7:- for real number α > 0, η and δ, the fractional operator, $\mathrm{I}_{0,z}^{\alpha,\eta,\delta}$ is defined by

$$\mathrm{I}_{0,z}^{\alpha,\eta,\delta}f(z)=\frac{z^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^z(z-t)^{\alpha-1}F\left(\alpha+\eta,-\delta;\alpha;1-\frac{t}{z}\right)f(t)dt, \tag{13}$$

Where f(z) is analytic function ina simply connected region containing the origin with order

$$f(z)=\ \mathrm{O}(|\mathrm{z}|^{\varepsilon}),z\to 0, where\ \varepsilon>\max(0,\eta-\delta)-1,$$

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n$$

and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & n = 0 \\ a(a+1)\ldots\ldots(a+n-1) & n \in N \end{cases}$$

and the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring log(z-t) to the real when z-t > 0. To prove our result we need lemma given by Srivastava [7].

Lemma1:- Let α > 0 and n > η – δ -1. Then

$$I_{0,z}^{\alpha,\eta,\delta} f(z) = \frac{\Gamma(n+1)\Gamma(n-\eta+\delta+1)}{\Gamma(n-\eta+1)\Gamma(n+\alpha+\delta+1)} z^{n-\eta}$$

making use of lemma1. We prove the following theorem.

Theorem4:- let η < 2, α >0, η(α + λ) ≤ 3 α and α + λ > -2. If f(z) be defined by the function (4) is in the class $X(\mathrm{a,b,c};\ \beta,\delta,\mathrm{k})$ then

$$\left|I_{0,z}^{\alpha,\eta,\lambda} f(z)\right| \geq \frac{\Gamma(2-\eta+\lambda)}{\Gamma(2-\eta)\Gamma(2+\alpha+\lambda)}\left(1 - \frac{2\beta(2-\eta+\lambda)(1-\delta)\Gamma(b)\Gamma(c+k)}{a(2-\eta)(2+\alpha+\lambda)\Gamma(c)\Gamma(b+k)}|z|\right) \qquad (14)$$

And

$$\left|I_{0,z}^{\alpha,\eta,\lambda} f(z)\right| \leq \frac{\Gamma(2-\eta+\lambda)}{\Gamma(2-\eta)\Gamma(2+\alpha+\lambda)}\left(1 + \frac{2\beta(2-\eta+\lambda)(1-\delta)\Gamma(b)\Gamma(c+k)}{a(2-\eta)(2+\alpha+\lambda)\Gamma(c)\Gamma(b+k)}|z|\right) \qquad (15)$$

for $z \in U_0$ where

$$U_0 = \begin{cases} U & n \leq 1 \\ U - \{0\} & n > 1 \end{cases}$$

The result is sharp for the function given by

$$f(z) = z - \frac{\beta(1-\delta)}{(\delta+\beta)} \cdot \frac{\Gamma(b)\Gamma(c+k)}{a\,\Gamma(c)\Gamma(b+k)} z^2$$

Proof:- By using above lemma, we have

$$I_{0,z}^{\alpha,\eta,\lambda} f(z) = \frac{\Gamma(2-\eta+\lambda)}{\Gamma(2-\eta)\Gamma(2+\alpha+\lambda)} z^{1-\eta} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\eta+\lambda+1)}{\Gamma(n-\eta+1)\Gamma(n+\alpha+\lambda+1)} \alpha_n z^{n-\eta} \qquad (16)$$

taking

$$H(z) = \frac{\Gamma(2-\eta)\Gamma(2+\alpha+\lambda)}{\Gamma(2-\eta+\lambda)} z^{\eta} I_{0,z}^{\alpha,\eta,\lambda} f(z) = z - \sum_{n=2}^{\infty} h(n)\, \alpha_n z^n$$

where

$$h(n) = \frac{(2-\eta+\lambda)_{n-1}(1)_n}{(2-\eta)_{n-1}(2+\alpha+\lambda)_{n-1}}, (n \geq 2) \qquad (17)$$

it can be easily seen that h(n) is non-increasing for n ≥ 2 and thus we have

$$0 < h(n) \leq h(2) = \frac{2(2-\eta+\lambda)}{(2-\eta)(2+\alpha+\lambda)} \qquad (18)$$

now by using theorem1 and (18), we have

$$|H(z)| \geq |z| - h(2)|z|^2 \sum_{n=2}^{\infty} \alpha_n \geq |z| - \frac{2\beta(2-\eta+\lambda)(1-\delta)\Gamma(b)\Gamma(c+k)}{a(2-\eta)(2+\alpha+\lambda)\Gamma(c)\Gamma(b+k)}|z|^2,$$

and

$$|H(z)| \leq |z| + h(2)|z|^2 \sum_{n=2}^{\infty} \alpha_n \leq |z| + \frac{2\beta(2-\eta+\lambda)(1-\delta)\Gamma(b)\Gamma(c+k)}{a(2-\eta)(2+\alpha+\lambda)\Gamma(c)\Gamma(b+k)} |z|^2,$$

This proves (14) and (15).

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